Arguing *Modus Tollens* with Conditional Probabilities in Place of Conditionals*

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Abstract

Arguing *Modus Tollens* with conditional probabilities in place of conditionals is generally invalid. There are, however, some simple premises that can be added to this form of argument to make it valid. Why these premises make this form of argument valid can be seen by observing the mathematical relationship between a conditional probability and its contrapositive conditional probability. Having a valid argument form for conditional probabilities that is like *Modus Tollens* may be useful when arguing for a negative, for example, arguing that certain unobserved things probably don't exist because, if they did exist, they probably would have been observed.

Keywords: Modus Tollens, conditional probability, contraposition

1 Introduction and Related Work

This is a discussion of arguments of a certain kind, which start with a premise about a conditional probability $\Pr(q|p)$ and conclude something about its contrapositive conditional probability $\Pr(\neg p|\neg q)$, which resemble *Modus Tollens*. The intent is to represent situations where one is attempting to infer the probability of $\neg p$ after learning $\neg q$, using one's prior expectations of q if p is true. Hence, conclusions are statements about the conditional probability $\Pr(\neg p|\neg q)$ and not the prior probability $\Pr(\neg p)$. The probability of the conditional is not discussed. Carl Wagner explained well why it is conditional probability and not the probability of a conditional that is of interest when investigating probabilistic versions of arguments.[1, Section 3.1].

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This form of argument has been discussed elsewhere. Elliott Sober criticized one such argument — an argument for Intelligent Design — showing that it is invalid.[2] Oaksford *et al.* argued on the basis of experimental results that $\Pr(\neg p|\neg q)$ correlates to how sure humans are of *Modus Tollens* inferences when presented with examples of arguments that involve different values for $\Pr(p)$, $\Pr(q)$ and $\Pr(q|p)$. This correlation suggests that this use of conditional probability is a model for the kind of *Modus Tollens* reasoning humans use intuitively.[3].

Other literature that combine probability and *Modus Tollens* incorporate probability into the argument differently. When Ernest Adams applied probability to contraposition (which, like *Modus Tollens*, involves the contrapositive of a conditional), he discussed the probability of a conditional in relation to the probability of its contrapositive, and not a conditional probability in relation to its contrapositive.[4, Sections 6.3 and 6.6] Wagner made the second premise in *Modus Tollens* an assertion about the prior probability of q and the conclusion an assertion about the prior probability of p.[1] Sobel followed Wagner and did the same.[5] Widaman asserted that, if the argument is to represent statistical testing of scientific hypotheses, $\neg q$ may be probable but not certain, but the relationship between the hypothesis p and its expected consequence q should be categorical and not merely probable.[6].

Mathematical assertions are made throughout the following sections. One theorem is stated. All can be verified with algebra and basic probability theory. In case proofs are wanted, proofs are provided in a section at the end.

2 Arguing *Modus Tollens* with Conditional Probability

One valid form of deductive argument is *Modus Tollens*, which goes like this:

·	$p \to q$	(If p, then q.)
·	$\neg q$	(Not q.)
$\therefore$	$\neg p$	(Therefore, not p.)

The first premise is a conditional; it asserts that, if the antecedent is true, the consequent is also true. But what if, in conditions where the antecedent is true, the consequent is not certain but merely probable? An argument one might try to make — one that I would be inclined to use in my own reasoning — is this:

· If p, then probably q.

· Not q.

$\therefore$ Probably not p.

which I would intend to mean

- $\Pr(q|p) \geq x$ (If p, then probably q.)
- $\neg q$ (Not q.)

$\therefore$ $\Pr(\neg p|\neg q) \geq x$ (Therefore, probably not p.)

where x is some minimum probability that I intend to argue for; it could be that the conclusion is almost certain, that it is more likely than not, or something in between. I would want to use it, for instance, to argue

- If Bigfoot existed, naturalists almost certainly would have identified a specimen by now[1](that is, found an animal or the remains of an animal that fits the description of Bigfoot and confirmed that it is not of a known species).
- Naturalists have not identified a specimen of Bigfoot.

$\therefore$ Therefore, Bigfoot almost certainly does not exist.

Unfortunately for me, this form of argument is invalid. Counterexamples are not hard to find. Several exist in published literature. One counterexample, stated by Pollard and Richardson[7] and repeated by Cohen[8], is

- If this person is an American, then it is very unlikely that this person is a member of Congress.
- This person is a member of Congress.

$\therefore$ It is very unlikely that this person is an American.

In this argument, the premises are logically consistent, but the conclusion is not just false, it is almost the opposite of the truth; if a person is a member of Congress (the United States' congress, that is), that person is almost certainly an American. As it turns out, the premises in this form of argument do not entail the conclusion. In fact, they do not entail anything about the probability of $\neg p$; Hailperin proved that for any value of $\Pr(q|p)$, it is possible for $\Pr(\neg p|\neg q)$ to be any value from 0 to 1, which is to say it could be any probability.[9, Theorem 5.42].

[1] With the first premise, I only intend to make a statement about my prior expectations. In spite of the problems raised by using the subjunctive mood, I must use it in the first premise because the second premise is part of my present knowledge. I take this argument to be an instance of the Problem of Old Evidence, and Howson's solution[10] applies; the conditional probability in the first premise is based a body of knowledge like my own, but with the fact that naturalists haven't found Bigfoot subtracted from it. While it isn't always possible to subtract a datum from a body of evidence and determine probabilities for the result, in this case I believe it is possible. In this case, I can ignore the observation and fall back on some generalities: all animals leave behind physical remains, and even elusive animals fail to avoid observation on some occasions, so there is always at least a small chance of such an animal providing evidence such as could be studied by naturalists; if a species of large land mammal exists, over time there must many chances for that evidence to occur, which combine to make it very probable that it would be discovered. Bigfoot is supposed to be a large land mammal. Therefore, I rate the probability of it being discovered as high, on the condition that it actually exists.

3 Adding Premises

So this kind of argument is generally invalid, but are there conditions where "if p, then probably q" and "not q" actually do entail "probably not p"? Put differently, can the argument be made valid by adding a premise to it? The answer is "yes". There is a general truth about the relationships between the probabilities of propositions, which is

Theorem 1. *For each comparison $\geq$, $>$, $=$, $<$ and $\leq$, for any propositions p and q such that $\Pr(p \wedge \neg q) > 0$, comparisons made between these pairs of probabilities are equivalent:*

$$\begin{aligned} \Pr(\neg q) &\text{ and } \Pr(p) \\ \Pr(\neg p) &\text{ and } \Pr(q) \\ \Pr(\neg p \wedge \neg q) &\text{ and } \Pr(p \wedge q) \\ \Pr(\neg p \vee \neg q) &\text{ and } \Pr(p \vee q) \\ \Pr(\neg p | \neg q) &\text{ and } \Pr(q | p) \end{aligned}$$

A consequence of this theorem is that adding any of these four equivalent premises will make the argument valid:[2]

$$\Pr(\neg q) \geq \Pr(p) \tag{P1}$$
$$\Pr(\neg p) \geq \Pr(q) \tag{P2}$$
$$\Pr(\neg p \wedge \neg q) \geq \Pr(p \wedge q) \tag{P3}$$
$$\Pr(\neg p \vee \neg q) \geq \Pr(p \vee q) \tag{P4}$$

Proof. Suppose $\Pr(q|p)$ is greater than or equal to x, and one of (P1), (P2), (P3) or (P4) is true. Then $\Pr(\neg p|\neg q)$ is greater than or equal to $\Pr(q|p)$, because, according to Theorem 1, it is equivalent to each of (P1), (P2), (P3) and (P4). Since $\Pr(\neg p|\neg q)$ is greater than or equal to $\Pr(q|p)$ and $\Pr(q|p)$ is greater than or equal to x, $\Pr(\neg p|\neg q)$ is also greater than or equal to x. □

4 A Conditional Probability and Its Contrapositive

To see why the addition of any of these premises makes the argument valid, it is helpful to examine the mathematical relationship between $\Pr(q|p)$ and $\Pr(\neg p|\neg q)$. This can be done by expressing $\Pr(\neg p|\neg q)$ as a function of $\Pr(q|p)$ and other quantities. There are many ways to do this[3], but if a ratio is considered to be a single quantity, then $\Pr(\neg p|\neg q)$ can be expressed as functions of $\Pr(q|p)$ and just one other quantity. It can be expressed as a function of $\Pr(q|p)$ and the ratio of $\Pr(p)$ to $\Pr(\neg q)$:

$$\Pr(\neg p|\neg q) = 1 - \frac{\Pr(p)}{\Pr(\neg q)}\left(1 - \Pr(q|p)\right) \tag{1}$$

[2]The fact that adding (P1) makes the argument valid has been observed by Eliot Sober.[2, footnote 14]

[3]For their purposes, Oaksford *et al.* expressed $\Pr(\neg p|\neg q)$ as a function of $\Pr(q|p)$ and two other quantities: $\Pr(p)$ and $\Pr(q)$.[3]

It can also be expressed as a function of $\Pr(q|p)$ and the ratio of $\Pr(p \wedge q)$ to $\Pr(\neg p \wedge \neg q)$:

$$\Pr(\neg p|\neg q) = \frac{1}{1 + \frac{\Pr(p \wedge q)}{\Pr(\neg p \wedge \neg q)} \left(\frac{1}{\Pr(q|p)} - 1 \right)} \tag{2}$$

From (1) and (2), a few things are clear: First, the larger $\Pr(q|p)$ is, the larger $\Pr(\neg p|\neg q)$ is. Next, the smaller the ratios $\Pr(p)\colon \Pr(\neg q)$ and $\Pr(q|p)\colon \Pr(\neg p|\neg q)$ are, the larger $\Pr(\neg p|\neg q)$ is. Last, if either of these ratios are 1, then $\Pr(\neg p|\neg q)$ is equal to $\Pr(q|p)$. So, if the intended conclusion of an argument is that $\Pr(\neg p|\neg q)$ is at least as probable as $\Pr(q|p)$, a premise that entails that one of these ratios is not more than 1:1 will be sufficient to complete it.

5 When the Probability of the First Premise Exceeds the Probability of the Conclusion

If $\Pr(q|p)$ is known to exceed the minimum value that is being argued for $\Pr(\neg p|\neg q)$, then the argument may be valid even if the aforementioned ratios are greater than 1. For instance, according to Equation (2), if, given p, we would be 99.9% sure of q, then we can be at least 99% sure of $\neg p$ given $\neg q$, even if the ratio of $\Pr(p \wedge q)$ to $\Pr(\neg p \wedge \neg q)$ is as large as 111:11. In general, if x is the minimum value that is being argued for $\Pr(\neg p|\neg q)$, and $\Pr(q|p)$ is greater than or equal to y, then the addition of either of these two premises is sufficient to make a valid argument:

$$\frac{\Pr(p)}{\Pr(\neg q)} \leq \frac{1 - x}{1 - y} \tag{P5}$$

$$\frac{\Pr(p \wedge q)}{\Pr(\neg p \wedge \neg q)} \leq \frac{y\,(1 - x)}{x\,(1 - y)} \tag{P6}$$

6 Valid Forms for Conditional Probability

Because of what has been discussed in the preceding sections, all of these forms of argument are valid:

$$\begin{array}{ll} \cdot & \Pr(q|p) \geq x \\ \cdot & \Pr(\neg q) \geq \Pr(p) \\ \hline \therefore & \Pr(\neg p|\neg q) \geq x \end{array}$$

$$\begin{array}{ll} \cdot & \Pr(q|p) \geq x \\ \cdot & \Pr(\neg p) \geq \Pr(q) \\ \hline \therefore & \Pr(\neg p|\neg q) \geq x \end{array}$$

$$
\begin{array}{ll}
\cdot & \Pr(q|p) \geq x \\
\cdot & \Pr(\neg p \wedge \neg q) \geq \Pr(p \wedge q) \\
\hline
\therefore & \Pr(\neg p|\neg q) \geq x
\end{array}
$$

$$
\begin{array}{ll}
\cdot & \Pr(q|p) \geq x \\
\cdot & \Pr(\neg p \vee \neg q) \geq \Pr(p \vee q) \\
\hline
\therefore & \Pr(\neg p|\neg q) \geq x
\end{array}
$$

$$
\begin{array}{ll}
\cdot & \Pr(q|p) \geq y \\
\cdot & \dfrac{\Pr(p)}{\Pr(\neg q)} \leq \dfrac{1-x}{1-y} \\
\hline
\therefore & \Pr(\neg p|\neg q) \geq x
\end{array}
$$

$$
\begin{array}{ll}
\cdot & \Pr(q|p) \geq y \\
\cdot & \dfrac{\Pr(p \wedge q)}{\Pr(\neg p \wedge \neg q)} \leq \dfrac{y\,(1-x)}{x\,(1-y)} \\
\hline
\therefore & \Pr(\neg p|\neg q) \geq x
\end{array}
$$

7 Examples of Use

The following are examples of how the ideas discussed in the preceding sections can be used. One shows why a counterexample to *Modus Tollens* that involves probability is invalid. The other makes a valid argument out of an invalid argument by taking additional information into account.

7.1 Pollard and Richardson's Counterexample

Let p be the meaning of "this person is an American" and q be the meaning of "this person is not a member of congress". Interpret "... is very unlikely" as $\Pr(\neg ...) > x$, where x is less than but close to 1. Then Pollard and Richardson's counterexample is an instance of arguing *Modus Tollens* with conditional probabilities in place of conditionals. The argument is not valid because it does not assert $\Pr(\neg q) \geq \Pr(p)$ nor anything equivalent, and it does not state how unlikely being a member of Congress is in relation to how unlikely it is to be American.

To explain why the argument leads to a false conclusion, consider the fact that members of Congress are required to be U.S. citizens. Let x be the total number of persons we are considering in this argument, whether human, extraterrestrial or other. Since non-citizens are not allowed to sit in Congress and the number of non-citizens in Congress has never been known to be more than zero, $\Pr(\neg p \wedge \neg q)$ is $y{:}x$, where y is some positive number less than 1. Since there are something on the order of three hundred million U.S. citizens and only a few hundred of them are members of Congress, $\Pr(p \wedge q)$ is about 300,000,000:x. Therefore, the ratio $\Pr(p \wedge q)\colon \Pr(\neg p \wedge \neg q)$ is about 300,000,000:y, which is a very large number. Condition (P3) is far from being met, and so the argument gives us no reason to think that the conclusion is true. On the contrary, it can be seen in (2) that, since $\Pr(q|p)$ is small (according to the first premise) and the ratio of $\Pr(p \wedge q)$ to $\Pr(\neg p \wedge \neg q)$ is large, $\Pr(\neg p|\neg q)$ is small, so there is a high probability that the person is an American.

7.2 Bigfoot Almost Certainly Does Not Exist

Let us return to the argument in Section 2 about the existence of Bigfoot. Let p be the meaning of "Bigfoot exists" and q be the meaning of "Naturalists have identified a specimen". As was stated before, simply arguing "if Bigfoot existed, naturalists almost certainly would have identified a specimen by now" won't be enough. To complete the argument, observe two things: First, it is extremely unlikely that naturalists would identify a specimen of Bigfoot if Bigfoot does not exist. It could only happen if naturalists (as a group) erroneously identified something as a specimen of Bigfoot. Naturalists are usually competent and most are not eager to confirm the existence of Bigfoot, so this is extremely unlikely. Second, most creatures that are commonly believed to be legendary are in fact legendary. The first observation implies that $\Pr(\neg p \wedge q)$ is very small. The second observation implies that the prior probability of Bigfoot's existence, $\Pr(p)$ is less than the prior probability of Bigfoot's nonexistence, $\Pr(\neg p)$. Taken together, we can be sure that the sum of $\Pr(p)$ and $\Pr(\neg p \wedge q)$ is less than $\Pr(\neg p)$. So the argument can be completed as

$$
\begin{array}{ll}
\cdot & \Pr(q|p) \geq x \\
\cdot & \Pr(\neg p \wedge q) + \Pr(p) < \Pr(\neg p) \\
\hline
\therefore & \Pr(\neg p|\neg q) \geq x
\end{array}
$$

This argument is valid because (P2) follows from its second premise. So Bigfoot almost certainly does not exist.

This argument's conclusion only answers a single cryptozoological question, but the argument serves to show how arguments of its kind may be of general use. It is sometimes said that "you can't prove a negative"[4], that is, you can't empirically prove that no objects of a kind exist, because it is impossible for humans to observe so much of the universe that they can be absolutely certain that nothing of the kind is anywhere. Sometimes, a person will argue for a negative by asserting it and insisting that, because "you can't prove a negative", the burden of proof is on the person asserting the positive. This may win debates, but it doesn't provide assurance that the negative is in fact true. If assurance is wanted, *Modus Tollens* with conditional probabilities in place of conditionals may be able to provide it. If there is a strong expectation that, if a thing existed, there would be evidence for it by now, and no

[4] Arguing against the existence of Bigfoot as an example of proving a negative has also been discussed by Hales[11]

evidence has been found, then, with some additional observations about prior probabilities, it might be established that its existence is unlikely, even placing a bound on the probability of its existence. In other words, this form of argument can be used to demonstrate that, in some situations, a failure to observe the positive proves that the positive, though possible, is very improbable.

8 Proofs

Proof of Theorem 1. A proof by cases. Before the cases, observe that, by multiplicative identity,

$$\frac{\Pr(p \wedge \neg q)}{\Pr(p \wedge \neg q)} \cdot \frac{\Pr(p)}{\Pr(\neg q)} = \frac{\Pr(p)}{\Pr(\neg q)}\,. \tag{3}$$

Then by (3) and the definition of conditional probability,

$$\frac{\Pr(p|\neg q)}{\Pr(\neg q|p)} = \frac{\Pr(p)}{\Pr(\neg q)}\,. \tag{4}$$

These inferences and the divisions in later steps of this proof are allowable because, by hypothesis, $\Pr(p \wedge \neg q) > 0$, and therefore $\Pr(p) > 0$ and $\Pr(\neg q) > 0$, so there will be no dividing by zero.

For the first case, suppose $\Pr(\neg q)$ greater than or equal to $\Pr(p)$, which is (P1). This first case will also prove that (P1), (P2), (P3) and (P4) are equivalent. By the Complement Rule, $\Pr(\neg q) \geq \Pr(p)$ is equivalent to

$$1 - \Pr(q) \geq 1 - \Pr(\neg p)\,,$$

which, by addition and cancelling, is equivalent to (P2):

$$\Pr(\neg p) \geq \Pr(q)\,,$$

which, by the Law of Total Probability, is equivalent to

$$\Pr(\neg p \wedge q) + \Pr(\neg p \wedge \neg q) \geq \Pr(p \wedge q) + \Pr(\neg p \wedge q)\,,$$

which, by subtraction, is equivalent to (P3):

$$\Pr(\neg p \wedge \neg q) \geq \Pr(p \wedge q)\,,$$

which, by DeMorgan's Law and the Complement Rule, is equivalent to

$$1 - \Pr(p \vee q) \geq 1 - \Pr(\neg p \vee \neg q)\,,$$

which, by addition and cancelling, is equivalent to (P4):

$$\Pr(\neg p \vee \neg q) \geq \Pr(p \vee q)\,.$$

By (P1) and division,

$$1 \geq \frac{\Pr(p)}{\Pr(\neg q)}\,. \tag{5}$$

and by (5), (4) and substitution

$$1 \geq \frac{\Pr(p|\neg q)}{\Pr(\neg q|p)}\,. \tag{6}$$

By (6) and multiplication,

$$\Pr(\neg q|p) \geq \Pr(p|\neg q)$$

and by the definition of complement,

$$1 - \Pr(q|p) \geq 1 - \Pr(\neg p|\neg q)$$

which, by addition and cancelling, is equivalent to

$$\Pr(\neg p|\neg q) \geq \Pr(q|p)\,.$$

For the other cases, follow an analogous line of reasoning while supposing $\Pr(p)$ is greater than $\Pr(\neg q)$, then equal, then less than, then less than or equal, and the proof will be complete. □

Proof of (1). Start with Bayes' Rule:

$$\Pr(\neg p|\neg q) = \frac{\Pr(\neg p \wedge \neg q)}{\Pr(\neg q)}\,.$$

By the Law of Total Probability:

$$\Pr(\neg p|\neg q) = \frac{\Pr(\neg q) - \Pr(p \wedge \neg q)}{\Pr(\neg q)}\,.$$

By the definition of conditional probability:

$$\Pr(\neg p|\neg q) = \frac{\Pr(\neg q) - \Pr(p)\Pr(\neg q|p)}{\Pr(\neg q)}\,. \tag{7}$$

As as aside: making two substitutions based on the Complement Rule yields the relation Oaksford *et al.* associated with *Modus Tollens* [3, Equation 4]:

$$\Pr(\neg p|\neg q) = \frac{1 - \Pr(q) - \Pr(p)\Pr(\neg q|p)}{1 - \Pr(q)} \,.$$

By (7) and the Complement Rule,

$$\Pr(\neg p|\neg q) = \frac{\Pr(\neg q) - \Pr(p)\Pr(\neg q|p)}{\Pr(\neg q)} \,.$$

By algebra,

$$\Pr(\neg p|\neg q) = 1 - \frac{\Pr(p)\Pr(\neg q|p)}{\Pr(\neg q)} \,.$$

By the Complement Rule,

$$\Pr(\neg p|\neg q) = 1 - \frac{\Pr(p)}{\Pr(\neg q)}\left(1 - \Pr(q|p)\right) \,.$$

$\square$

Proof of (2). First, express $\Pr(p \wedge \neg q)$ in terms of $\Pr(q|p)$. By the definition of conditional probability and the Law of Total Probability,

$$\Pr(q|p) = \frac{\Pr(p \wedge q)}{\Pr(p \wedge q) + \Pr(p \wedge \neg q)} \,. \tag{8}$$

Rearranging (8) algebraically yields

$$\Pr(p \wedge \neg q) = \Pr(p \wedge q)\left(\frac{1}{\Pr(q|p)} - 1\right) \tag{9}$$

The relationship of $\Pr(p \wedge \neg q)$ to $\Pr(\neg p|\neg q)$ can be found by an analogous line of reasoning. By the definition of conditional probability and the Law of Total Probability,

$$\Pr(\neg p|\neg q) = \frac{\Pr(\neg p \wedge \neg q)}{\Pr(p \wedge \neg q) + \Pr(\neg p \wedge \neg q)} \,. \tag{10}$$

Rearranging (10) algebraically yields

$$\Pr(p \wedge \neg q) = \Pr(\neg p \wedge \neg q)\left(\frac{1}{\Pr(\neg p|\neg q)} - 1\right) \,. \tag{11}$$

Now combine the two relations. By (9), (11) and the transitivity of equality,

$$\Pr(p \wedge q)\left(\frac{1}{\Pr(q|p)} - 1\right) = \Pr(\neg p \wedge \neg q)\left(\frac{1}{\Pr(\neg p|\neg q)} - 1\right). \tag{12}$$

and finally, rearranging (12) with algebra yield

$$\Pr(\neg p|\neg q) = \frac{1}{1 + \frac{\Pr(p \wedge q)}{\Pr(\neg p \wedge \neg q)}\left(\frac{1}{\Pr(q|p)} - 1\right)}.$$

□

Proof that (P5) *with* $\Pr(q|p) \geq y$ *entail* $\Pr(\neg p|\neg q) \geq x$. Suppose (P5) is true:

$$\frac{\Pr(p)}{\Pr(\neg q)} \leq \frac{1-x}{1-y}.$$

Rearranging this inequation with algebra yields

$$1 + \frac{\Pr(\neg q)}{\Pr(p)}(x-1) \leq y. \tag{13}$$

Suppose $\Pr(q|p)$ is greater than or equal to y. Since "greater than or equal to" is transitive,

$$1 + \frac{\Pr(\neg q)}{\Pr(p)}(x-1) \leq \Pr(q|p)$$

can be inferred from (13) and $\Pr(\neg p|\neg q) \geq x$. Rearranging this inequation yields

$$1 - \frac{\Pr(p)}{\Pr(\neg q)}(1 - \Pr(q|p)) \geq x$$

and by (1),

$$\Pr(\neg p|\neg q) \geq x.$$

□

Proof that (P6) *with* $\Pr(q|p) \geq y$ *entail* $\Pr(\neg p|\neg q) \geq x$. Suppose (P6) is true:

$$\frac{\Pr(p \wedge q)}{\Pr(\neg p \wedge \neg q)} \leq \frac{y(1-x)}{x(1-y)}.$$

Rearranging this inequation with algebra yields

$$\frac{1}{1 + \frac{\Pr(\neg p \wedge \neg q)}{\Pr(p \wedge q)} \cdot \frac{1-x}{x}} \leq y \,. \tag{14}$$

Suppose $\Pr(q|p)$ is greater than or equal to y. Since "greater than or equal to" is transitive,

$$\frac{1}{1 + \frac{\Pr(\neg p \wedge \neg q)}{\Pr(p \wedge q)} \cdot \frac{1-x}{x}} \leq \Pr(q|p)$$

can be inferred from (14) and $\Pr(q|p) \geq y$. Rearranging this inequation yields

$$\frac{1}{1 + \frac{\Pr(p \wedge q)}{\Pr(\neg p \wedge \neg q)} \left(\frac{1}{\Pr(q|p)} - 1 \right)} \geq x$$

and by (2),

$$\Pr(\neg p|\neg q) \geq x \,.$$

□

Proof that (P2) follows from the second premise of the Bigfoot argument. The second premise of the Bigfoot argument is

$$\Pr(\neg p) > \Pr(\neg p \wedge q) + \Pr(p) \,.$$

Expanding $\Pr(p)$ according to the Law of Total Probability yields

$$\Pr(\neg p) > \Pr(\neg p \wedge q) + \Pr(p \wedge q) + \Pr(p \wedge \neg q) \,.$$

Combining terms to make $\Pr(q)$ according to the Law of Total Probability yields

$$\Pr(\neg p) > \Pr(q) + \Pr(p \wedge \neg q) \,.$$

Since $\Pr(p \wedge \neg q)$ is a probability, it must be greater than or equal to zero, and therefore

$$\Pr(\neg p) > \Pr(q) \,.$$

"Greater than" implies "greater than or equal", so

$$\Pr(\neg p) \geq \Pr(q)\,.$$

$\square$

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