PROBABILITY IN LIGHT OF INDEPENDENT TESTIMONY FOR MUTUALLY EXCLUSIVE POSSIBILITIES

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ABSTRACT. In this document, Bayes' Rule is used to find the probability of a proposition in light of multiple testimonies that may conflict with each other or corroborate each other, with the constraint that the testimonies are independent of one another and each attest to one of a set of mutually exclusive alternatives. The probability of the proposition is shown to be a function of the prior probabilities of the alternatives and the probability that the witnesses would claim what they have claimed given each of the alternatives. For cases where the probability of the witness telling the truth is known and the witness' truthfulness is independent of claims, theorems are stated which allow the probability of a witness telling the truth to be substituted for the probability that the witness would claim what they have claimed given an alternative. The theorems stated are used to solve some example problems, including a problem stated by Augustus de Morgan in *Formal Logic*.

1. Introduction

There are many factors that complicate the use of testimony to estimate the probability of a proposition. Testimonies may corroborate or contradict each other. One witness may be more or less likely to tell the truth than another. Witnesses may be biased for or against making certain claims. The claims may be more or less likely *a priori*. Estimating probability in light of ths kind of evidence may not always be feasible, but if the testimony is independent and it is limited to a set of mutually exclusive alternatives, there is a formula that can take all of these things into account.

The discussion proceeds as follows: The problem is stated. The basic solution and a corollary are given; both relate the probability of a proposition to the probabilities of witnesses making certain claims and the prior probabilities of the things that are claimed. To extend the solution's applicability, some additional theorems are stated that relate the probabilities of a witness making claims to the probability that the witness would tell the truth. To illustrate how these formulas and theorems might be used, example problems are presented and solved. In case it is helpful, some notation is explained and axioms, a lemma and a definition are stated. At the end of the discussion, theorems stated and assertions made during the discussion, which require proof, are proven from standard axioms and definitions.

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2. Stating the Problem

Let A be a set of alternative possibilities. Let W be a set of witnesses that have each attested to exactly one of the alternatives in A. Let q be one of the alternatives in A. Let c be a predicate on two variables such that "c_w^a" asserts "witness w claims that alternative a is true". Let T be a function that maps each witness in W to the alternative in A to which the witness attested. Then the probability of q given the testimony of W is

$$\Pr\left(q \middle| \bigwedge_{w \in W} c_w^{T(w)}\right),$$

which may be read as "the probability of q given the fact that the witnesses in W claim what they have claimed". This is the quantity that is being sought. This quantity should be deduced from these quantities, which might be known or estimated:

- ◇ The prior probabilities of the alternatives that the witnesses might have attested to
- ◇ The probabilities that witnesses would make their claims, given each of the alternatives
- ◇ The probability that each witness would tell the truth

The problem is limited to situations which meet all of these conditions:

1. The claims are constrained to a set of alternatives that are mutually exclusive of one another.
2. The alternatives cover all possibilities.
3. None of the alternatives are known *a priori* to be false.
4. The witnesses' claims are conditionally independent of one another given each of the alternatives they might have claimed.

Note that the claims are to be *conditionally* independent. In general, it would not be expected or desirable for the claims to be unconditionally independent. If two witnesses are inclined to tell the truth, then if one makes a certain claim, it raises the probability that the claim is true, which in turn raises the probability that the other witness makes the same claim. Conditional independence means one witness's claim may be influenced by many things (hopefully including reality and a will to tell the truth), but not by the other witnesses' claims.

3. The Basic Solution and a Corollary

Under the conditions stated above, the basic solution is a formula that can be derived from Bayes' Rule, the Law of Total Probability and the definition of conditional independence. It has already been stated and proven by Hu and Qu as Theorem 4 in "Bayes' Theorem under Conditional Indepenence"[1].

Theorem 1 (Hu and Qu's Theorem 4)**.** *If the alternatives in A are mutually exclusive and cover all possibilities, and none are known* a priori *to be false, and all the claims of W are all conditionally independent of one another given any of* *these alternatives, then*

$$\Pr\left(q\middle|\bigwedge_{w\in W}c_w^{T(w)}\right)=\frac{\Pr(q)\prod_{w\in W}\Pr\left(c_w^{T(w)}\middle|q\right)}{\sum_{a\in A}\Pr(a)\prod_{w\in W}\Pr\left(c_w^{T(w)}\middle|a\right)}\,.$$

A corollary that relates the probability of one alternative to another given the claims of witnesses follows straightforwardly from Theorem 1:

Corollary 1. *If the alternatives in A are mutually exclusive and cover all possibilities, and none are known* a priori *to be false, and all the claims of W are all conditionally independent of one another given any of these alternatives, then the ratio of the probabilities for any two alternatives q and r in A given the claims of W is*

$$\frac{\Pr\left(q\middle|\bigwedge_{w\in W}c_w^{T(w)}\right)}{\Pr\left(r\middle|\bigwedge_{w\in W}c_w^{T(w)}\right)}=\frac{\Pr(q)\prod_{w\in W}\Pr\left(c_w^{T(w)}\middle|q\right)}{\Pr(r)\prod_{w\in W}\Pr\left(c_w^{T(w)}\middle|r\right)}\,.$$

4. Equivalences

Let t be a predicate on one variable such that "t_w" asserts "witness w is telling the truth". If so, then the predicates c and t are related logically. For instance, if ac_w^a, then t_w (if alternative a is true and w claims a, then w is telling the truth). The theorems in this section state some of these relations. Both theorems follow straightforwardly from what has been postulated about c and t.

Theorem 2. *These equivalences are true for all propositions a and b and any witness w who has attested to a or b:*

$$ac_w^a t_w \equiv ac_w^a\,, \tag{E1}$$
$$ac_w^a t_w \equiv at_w\,, \tag{E2}$$
$$ac_w^a t_w \equiv c_w^a t_w\,, \tag{E3}$$
$$\bar{a}c_w^a \bar{t}_w \equiv \bar{a}c_w^a\,, \tag{E4}$$
$$\bar{a}c_w^a \bar{t}_w \equiv c_w^a \bar{t}_w\,, \tag{E5}$$
$$a\bar{c}_w^a \bar{t}_w \equiv a\bar{c}_w^a\,, \tag{E6}$$
$$a\bar{c}_w^a \bar{t}_w \equiv a\bar{t}_w\,,\ \textit{and} \tag{E7}$$
$$ac_w^b \bar{t}_w \equiv ac_w^b \qquad \textit{if} \quad a\rightarrow\bar{b}\,. \tag{E8}$$

Additional equivalences are true in circumstances where there are exactly two alternatives that a witness may attest to, e.g., "The accused committed the crime" and "the accused did not commit the crime".

Theorem 3. *If A is a set of alternatives, a is a member of A and w is a witness that has attested to one of the alternatives in A, then these equivalences are all true:*

$$\bar{a}\bar{c}_w^a t_w \equiv \bar{a}\bar{c}_w^a \quad \textit{if} \quad |A| = 2\,, \tag{E9}$$

$$\bar{a}\bar{c}_w^a t_w \equiv \bar{a} t_w \quad \textit{if} \quad |A| = 2\,, \tag{E10}$$

$$\bar{a}\bar{c}_w^a t_w \equiv \bar{c}_w^a t_w \quad \textit{if} \quad |A| = 2\,, \tag{E11}$$

$$a\bar{c}_w^a \bar{t}_w \equiv \bar{c}_w^a \bar{t}_w \quad \textit{if} \quad |A| = 2\,, \tag{E12}$$

$$\bar{a} c_w^a \bar{t}_w \equiv \bar{a}\bar{t}_w \quad \textit{if} \quad |A| = 2\,, \tag{E13}$$

$$\bar{c}_w^a \equiv c_w^{\bar{a}} \quad \textit{if} \quad |A| = 2\,, \textit{ and} \tag{E14}$$

$$\bar{c}_w^{\bar{a}} \equiv c_w^a \quad \textit{if} \quad |A| = 2\,. \tag{E15}$$

5. Some Substitutes for $\Pr\left(c_w^{T(w)} \middle| a\right)$

The theorems in this section are true for all a in A and all w in W where A is a set of alternatives and W is a set of witnesses who have each attested to one of the alternatives in A. These theorems identify substitutes for $\Pr\left(c_w^{T(w)} \middle| a\right)$, which appears in Theorem 1 and Corollary 1.

Theorem 4. *The probability of a witness claiming an alternative when that alternative is true is equal to the probability of the witness telling the truth;*

$$\Pr\left(c_w^a | a\right) = \Pr\left(t_w | a\right)\ .$$

Theorem 5. *If a witness's truthfulness is independent of an alternative, then the probability of the witness claiming that alternative when the alternative is true is equal to the prior probability of the witness telling the truth;*

$$\Pr\left(c_w^a | a\right) = \Pr\left(t_w\right) \quad \textit{if} \quad a \perp\!\!\!\perp t_w\ .$$

Theorem 6. *Given an alternative, if a witness, when not telling the truth, claims another alternative in proportion to its prior probability, then if the witness's truthfulness is independent of the given alternative being true, the probability that the witness claims the other alternative is equal to the product of the witness not telling the truth and the proportion of the prior probability of the other alternative to the prior probabilities of all of the alternatives that are not the given alternative;*

$$\Pr\left(c_w^b \middle| a\right) = \frac{\Pr\left(\bar{t}_w\right)\Pr\left(b\right)}{\Pr\left(\bar{a}\right)} \quad \textit{if} \quad a \perp\!\!\!\perp \bar{t}_w \textit{ and } \Pr\left(c_w^b \middle| a\bar{t}_w\right) = \frac{\Pr\left(b\right)}{\Pr\left(\bar{a}\right)} \textit{ and } b \to \bar{a}\,.$$

Theorem 6 applies to situations where a witness does not exhibit bias for or against the alternative that they falsely claimed; the claim was made as if it was the result of a random selection among the false alternatives.

Theorem 7. *If there are only two alternatives and a witness's truthfulness is independent of them, then the probability of a witness claiming the false alternative is equal to the probability of the witness not telling the truth;*

$$\Pr\left(c_w^a | \bar{a}\right) = \Pr\left(\bar{t}_w\right) \quad \textit{if} \quad |A| = 2 \textit{ and } a \perp\!\!\!\perp t_w\ .$$

6. Example Problems

Example 1 (Dishonest accusations)**.** Alice, Bob and Dan attempt to rob a jewelry store. The only other person present is a clerk, whom one of them threatens with a pistol. Things do not go well and the clerk is shot dead. Alice, Bob and Dan are the only living witnesses to the event. There is no physical evidence to indicate which of them is the shooter. The police apprehend the three and interrogate each of them separately. During the interrogations, each makes an accusation. None of the suspects are willing to go to jail and none are loyal to any of the others. The suspects did not have time to confer beforehand and no plea bargains were offered in such a way as to motivate a false confession. Because of all this, they are expected to testify as follows: The guilty one will almost certainly not confess and will instead accuse one of the others, with equal probability of accusing either. The others, who have no known motive to protect the guilty one, will almost certainly accuse the guilty one. Let m_x be a predicate of one variable that asserts that x is the murderer. Let $W = \{Alice, Bob, Dan\}$, $a \equiv m_{Alice}$, $b \equiv m_{Bob}$, $d \equiv m_{Dan}$ and $A = \{a, b, d\}$. Let s in the following equations represent small positive numbers that are not necessarily equal to one another. Under these circumstances, the probabilities $\Pr\left(c_w^{T(w)} \middle| m_x\right)$ for x and w in W are as follows:

If a witness is not the murderer, the probability that he or she accuses the murderer is high;

$$\Pr\left(c_w^{T(w)} \middle| m_x\right) = 1 - s \qquad \text{if} \quad x \neq w \text{ and } m_x \equiv T(w)\,. \tag{1}$$

If a witness is not the murderer, the probability that he or she makes a false accusation is low and it is divided evenly amongst the suspects who are not the murderer;

$$\Pr\left(c_w^{T(w)} \middle| m_x\right) = \frac{s}{|A| - 1} \quad \text{if} \quad x \neq w \text{ and } m_x \not\equiv T(w)\,. \tag{2}$$

If a witness is the murderer, the probability that he or she confesses (and therefore accuses the murderer) is low;

$$\Pr\left(c_w^{T(w)} \middle| m_x\right) = s \qquad \text{if} \quad x = w \text{ and } m_x \equiv T(w)\,. \tag{3}$$

If a witness is the murderer, the probability that he or she makes a false accusation is high and it is divided evenly amongst the other suspects;

$$\Pr\left(c_w^{T(w)} \middle| m_x\right) = \frac{1 - s}{|A| - 1} \quad \text{if} \quad x = w \text{ and } m_x \not\equiv T(w)\,. \tag{4}$$

If, prior to the accusations, the probabilities that one suspect or another committed the murder are all equal, then the approximate probability of any one of the three being the murderer can be calculated from Theorem 1. In most circumstances such as this, no suspect confesses and two of the suspects accuse the same person; there is corroboration between two of the witnesses and it is almost certain that the suspect whom they accuse is the murderer. In the other cases, the probability of a suspect being the murderer is not so obvious. In the case where one suspect confesses and the other two accuse each other, one might think that the suspect who confessed is the murderer, because people usually do not confess to crimes that they did not commit, and the other two witnesses' accusations cancel each other out. But actually, the one who confessed is probably *not* the murderer. Suppose Alice confesses, Bob accuses Dan, and Dan accuses Bob. Given these claims and Corollary 1, the ratio of the probability of Alice being the murderer to the probability of Bob being the murderer is

$$\frac{\Pr\left(a \middle| \bigwedge_{w \in W} c_w^{T(w)}\right)}{\Pr\left(b \middle| \bigwedge_{w \in W} c_w^{T(w)}\right)} = \frac{\Pr(a)\Pr\left(c^a_{Alice}|a\right)\Pr\left(c^d_{Bob}\middle|a\right)\Pr\left(c^b_{Dan}\middle|a\right)}{\Pr(b)\Pr\left(c^a_{Alice}|b\right)\Pr\left(c^d_{Bob}\middle|b\right)\Pr\left(c^b_{Dan}\middle|b\right)}. \tag{5}$$

Making the assumption that $\Pr(a)$ and $\Pr(b)$ are equal and using (1), (2), (3) and (4) to find appropriate substitions for the other terms in (5) yields

$$\frac{\Pr\left(a \middle| \bigwedge_{w \in W} c_w^{T(w)}\right)}{\Pr\left(b \middle| \bigwedge_{w \in W} c_w^{T(w)}\right)} = \frac{s_1 \dfrac{s_2}{|A|-1} \cdot \dfrac{s_3}{|A|-1}}{\dfrac{s_4}{|A|-1} \cdot \dfrac{1-s_5}{|A|-1}(1-s_6)} = \frac{s_1 s_2 s_3}{s_4(1-s_5)(1-s_6)}.$$

As can be seen, the probability that Alice is the murderer given these accusations is the product of three small numbers, but the probability that Bob is the murderer given these accusations is the product of one small number and two instances of one minus a small number. Unless some of these small numbers differ by many orders of magnitude, the probability that Alice is the murderer is much smaller than the probability that Bob is the murderer. By the same line of reasoning, the ratio of the probability of Alice being the murderer to Dan being the murderer is also small. Therefore, although Alice confessed, she is probably not the murderer. This can be explained by noting that if Alice is the murderer, three unlikely events have occurred, but if Bob or Dan is the murderer, then only one unlikely event has occurred.

Example 2 (Unreliable but unbiased witnesses to a roll of a fair die)**.** Suppose someone rolls a fair, six-sided die. There is a witness and this witness is very unreliable; he misreports half of the rolls that he witnesses. However, he is unbiased in the sense that he is not more likely to be wrong about one number than another and, when he is wrong, he does not choose one number more often than another. He says that a six was rolled. How does his testimony affect the probability that a six was rolled? Is the posterior probability the same because what he says is as often as not false? What if another, similarly unbiased but even less reliable witness, who mispreports two thirds of the rolls he witnesses, says it is a six? Does his testimony decrease the probability that a six was rolled because he is more often than not wrong?

To answer these questions, start with Theorem 1. In this scenario, A is the set of possible rolls, one through six, q is the event that a six was rolled and W are the two unreliable witnesses. Since each roll is equally probable, the priors in Theorem 1 cancel out. Since the witnesses are no more or less likely to tell the truth when a six is rolled, Theorem 5 allows the terms that represent the probability that they would claim a six when a six was rolled to be replaced with $\Pr(t_w)$. Since neither of the witnesses are biased, Theorem 6 can be applied to the terms that represent the probability that they would claim a six when a number other than a six was rolled. The result of these substitutions is

$$\Pr\left(q \middle| \bigwedge_{w\in W} c_w^q\right) = \frac{\prod_{w\in W} \Pr(t_w)}{\prod_{w\in W} \Pr(t_w) + \sum_{a\in A\setminus\{q\}} \prod_{w\in W} \frac{\Pr(\bar{t}_w)\Pr(a)}{\Pr(\bar{q})}}. \tag{6}$$

Since the prior probabilities of all rolls are equal, this can be simplified to be

$$\Pr\left(q \middle| \bigwedge_{w\in W} c_w^q\right) = \frac{\prod_{w\in W} \Pr(t_w)}{\prod_{w\in W} \Pr(t_w) + (|A|-1)^{1-|W|} \prod_{w\in W} \Pr(\bar{t}_w)}. \tag{7}$$

Considering only the testimony of the first witness, the probability that a six was rolled is $\frac{1}{2}$;

$$\Pr\left(q \middle| \bigwedge_{w\in W} c_w^q\right) = \frac{\frac{1}{2}}{\frac{1}{2} + (6-1)^{1-1}\left(1-\frac{1}{2}\right)} = \frac{1}{2}.$$

The first witness's testimony increases the probability that a six was rolled because, though he does not tell the truth more often than not, he is not biased and his assertions are true more often than a random guess. If the testimony of both witnesses is taken into account, the probability is $\frac{5}{7}$;

$$\Pr\left(q \middle| \bigwedge_{w\in W} c_w^q\right) = \frac{\frac{1}{2}\cdot\frac{1}{3}}{\frac{1}{2}\cdot\frac{1}{3} + (6-1)^{2-1}\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)} = \frac{5}{7}.$$

The second witness's testimony, though less reliable than that of the first witness, is still better than a random guess, so it raises the probability of a six even further. The corroboration of the two witnesses raises the probability into the range of more-likely-than-not.

Example 3 (Independent testimony, according to de Morgan)**.** In chapter X of *Formal Logic*, titled "On Probable Inference"[2], Augustus de Morgan discusses several problems having to do with "independent testimonies to the truth of an assertion". For the first problem, he presents some formulae as solutions for certain circumstances. The first formula is a solution to circumstances where there are several independent witnesses for a single assertion and the probability of each telling the truth is known. He accounts for the prior probability of the assertion by including, in his words, "the initial testimony of the mind itself which is to form the judgement" as one of the testimonies. The formula, expressed in the notation of the present discussion, is

$$\Pr\left(q \middle| \bigwedge_{w\in W} c_w^q\right) = \frac{\Pr(q)\prod_{w\in W} \Pr(t_w)}{\Pr(q)\prod_{w\in W} \Pr(t_w) + \Pr(\bar{q})\prod_{w\in W} \Pr(\bar{t}_w)}. \tag{8}$$

He apparently intends the testimonies to be independent in the sense that the probabilities of each being true are independent of one another. Also, because the prior estimate of probability counts as testimony, the probability of each testimony being true is independent of the prior probability of the assertion. In symbolic form, this independence is

$$\bigwedge_{w\in W}\bigwedge_{x\in W} t_w \perp\!\!\!\perp t_x \text{ and } \bigwedge_{w\in W} q \perp\!\!\!\perp t_w\,, \tag{9}$$

from which the formula can be derived straightforwardly. Interestingly, the same formula can be derived from Theorem 1 if the testimonies are independent in a different sense, where the witnesses are conditionally independent of one another given each possible alternative, and the witnesses are unbiased in the sense that the probability of each witnesses telling the truth is independent of the probability of the assertion. In symbolic form, this is

$$\bigwedge_{a\in A}\left(\prod_{w\in W}\Pr\left(c_w^a|a\right) = \Pr\left(\bigwedge_{w\in W} c_w^{T(w)}\middle| a\right)\right) \text{ and } \bigwedge_{w\in W} q \perp\!\!\!\perp t_w\,. \tag{10}$$

The formula follows from both pairs of assumptions even though they are not equivalent – (9) does not entail (10) and (10) does not entail (9).

The second formula is an equation of two ratios, expressed as an analogy, that applies to cases where threre is independent testimony for and against an assertion and the credibility of all witnesses are equal. The formula, expressed in the notation of the present discussion, with m being the number of testimonies in favor of an assertion q and n being the number of testimonies against q, is

$$\frac{\Pr\left(q\middle|\bigwedge_{w\in W} c_w^{T(w)}\right)}{\Pr\left(\bar{q}\middle|\bigwedge_{w\in W} c_w^{T(w)}\right)} = \frac{\Pr\left(t_w\right)^m\left(1-\Pr\left(t_w\right)\right)^n}{\Pr\left(t_w\right)^n\left(1-\Pr\left(t_w\right)\right)^m} = \frac{\Pr\left(t_w\right)^{m-n}}{\left(1-\Pr\left(t_w\right)\right)^{m-n}}\,. \tag{11}$$

As with (8), de Morgan inferred it from (9). He did not, however, consider the prior probability of the assertion. By omitting it from the formula, he effectively assumed that $\Pr(q) = \Pr(\bar{q})$. Equation (11) can also be derived from Corollary 1 if (10) is assumed and $\Pr(q) = \Pr(\bar{q})$.

7. Notation, Axioms and a Definition

(There is nothing novel in this section; it is included to make the intended meanings of symbols and terms explicit and to show that this discussion is grounded in modern probability theory).

Throughout this discussion, an expression of the form "pq", where p and q are Boolean expressions, signifies a logical conjunction of the two, i.e. "p AND q". A line over a symbol denotes negation, e.g. "$\bar{c}_w^a$" asserts "witness w did not claim a". An expression of the form "$\bigwedge_{p\in P}\phi$" asserts that the expression ϕ is true for each member of P. An expression of the form "$\bigvee_{p\in P}\phi$" asserts that the expression ϕ is true for some member of P. An expression of the form "$p \perp\!\!\!\perp q$" asserts that the probabilities of p and q are independent. An expression of the form "$\prod_{x\in X} f(x)$" denotes the result of applying function f to each member of X and multiplying the results together. An expression of the form "$\sum_{x\in X} f(x)$" denotes the result of applying function f to each member of X and adding the results together.

The Kolmogorov Axioms, stated in terms of a set of mutually exclusive propositions P such that $\bigvee_{p\in P} p$, are

$$\Pr(p) \geq 0 \qquad \text{if} \quad p \in P\,, \tag{K1}$$

$$\Pr\left(\bigvee_{p\in P} p\right) = 1\,,\ \text{and} \tag{K2}$$

$$\Pr\left(\bigvee_{p\in S} p\right) = \sum_{p\in S} \Pr(p) \quad \text{if} \quad S \subseteq P\,. \tag{K3}$$

This consequence of (K2) and (K3) is useful for combining probability with logic:

$$\Pr(p) = \Pr(q) \qquad \text{if} \quad p \equiv q\,. \tag{K4}$$

It will be invoked several times in the following proofs.

Definition 1 (Conditional independence)**.** Propositions p and q are *conditionally independent* given proposition r if and only if

$$\Pr(pq|r) = \Pr(p|r)\Pr(q|r)\ .$$

8. Proofs

Proof of (K4).

$$p \vee \bar{p} \qquad \text{by excluded middle.} \tag{12}$$

$$\Pr(p \vee \bar{p}) = 1 \qquad \text{by (K2), (12).} \tag{13}$$

$$\Pr(p) + \Pr(\bar{p}) = 1 \qquad \text{by (K3), (13).} \tag{14}$$

Suppose $q \equiv p$. Then

$$q \vee \bar{p} \qquad \text{by hypothesis, (12), substitution.} \tag{15}$$

$$\Pr(q \vee \bar{p}) = 1 \qquad \text{by (K2), (15).} \tag{16}$$

$$\Pr(q) + \Pr(\bar{p}) = 1 \qquad \text{by (K3), (16).} \tag{17}$$

$$\Pr(p) + \Pr(\bar{p}) = \Pr(q) + \Pr(\bar{p}) \qquad \text{by (14), (17), transitivity.} \tag{18}$$

$$\Pr(p) = \Pr(q) \qquad \text{by (18), cancelling out.}$$

Therefore

$$\Pr(p) = \Pr(q) \quad \text{if} \quad p \equiv q\,.$$

□

Proof of Theorem 1. Consider the probability of an alternative q that is in A. According to Bayes' Rule,

$$\Pr\left(q \middle| \bigwedge_{w\in W} c_w^{T(w)}\right) = \frac{\Pr(q)\Pr\left(\bigwedge_{w\in W} c_w^{T(w)} \middle| q\right)}{\Pr\left(\bigwedge_{w\in W} c_w^{T(w)}\right)}\,. \tag{19}$$

Applying the Law of Total Probability to the denominator on the right-hand side of (19) yields

$$\Pr\left(q \middle| \bigwedge_{w\in W} c_w^{T(w)}\right) = \frac{\Pr(q)\Pr\left(\bigwedge_{w\in W} c_w^{T(w)} \middle| q\right)}{\sum_{a\in A}\Pr(a)\Pr\left(\bigwedge_{w\in W} c_w^{T(w)} \middle| a\right)}. \tag{20}$$

Since the claims made by members of W are independent given any of the alternatives in A, Theorem 1 follows from (20) and the definition of conditional independence:

$$\Pr\left(q \middle| \bigwedge_{w\in W} c_w^{T(w)}\right) = \frac{\Pr(q)\prod_{w\in W}\Pr\left(c_w^{T(w)} \middle| q\right)}{\sum_{a\in A}\Pr(a)\prod_{w\in W}\Pr\left(c_w^{T(w)} \middle| a\right)}.$$

□

Proof of Theorem 4.

$$\Pr(c_w^a|a) = \frac{\Pr(ac_w^a)}{\Pr(a)} \quad \text{by conditional probability.} \tag{21}$$

$$\frac{\Pr(ac_w^a)}{\Pr(a)} = \frac{\Pr(at_w)}{\Pr(a)} \quad \text{by (E1), (E2), (K4).} \tag{22}$$

$$\frac{\Pr(at_w)}{\Pr(a)} = \Pr(t_w|a) \quad \text{by conditional probability.} \tag{23}$$

By equations (21), (22), (23) and the transitive property of equality,

$$\Pr(c_w^a|a) = \Pr(t_w|a)\ .$$

□

Proof of Theorem 5. Suppose $a \perp\!\!\!\perp t_w$. Then

$$\Pr(t_w|a) = \Pr(t_w) \quad \text{by hypothesis.} \tag{24}$$

$$\Pr(c_w^a|a) = \Pr(t_w) \quad \text{by Theorem 4, (24), transitivity.}$$

Therefore

$$\Pr(c_w^a|a) = \Pr(t_w) \quad \text{if} \quad a \perp\!\!\!\perp t_w\ .$$

□

Proof of Theorem 6. Suppose $a \perp\!\!\!\perp \bar{t}_w$ and $\Pr\left(c_w^b | a\bar{t}_w\right) = \frac{\Pr(b)}{\Pr(\bar{a})}$ and $b \to \bar{a}$. Then

$$\frac{\Pr(b)}{\Pr(\bar{a})} = \Pr\left(c_w^b \middle| a\bar{t}_w\right) \quad \text{by hypothesis.} \tag{25}$$

$$= \frac{\Pr\left(ac_w^b\bar{t}_w\right)}{\Pr\left(a\bar{t}_w\right)} \quad \text{by conditional probability.} \tag{26}$$

$$= \frac{\Pr\left(ac_w^b\right)}{\Pr\left(a\bar{t}_w\right)} \quad \text{by (E8), hypothesis, \textit{modus tollens}.} \tag{27}$$

$$= \frac{\Pr\left(ac_w^b\right)}{\Pr(a)\Pr\left(\bar{t}_w\right)} \quad \text{by hypothesis, independence.} \tag{28}$$

$$= \frac{\Pr\left(c_w^b \middle| a\right)}{\Pr\left(\bar{t}_w\right)} \quad \text{by conditional probability.} \tag{29}$$

$$\frac{\Pr(b)}{\Pr(\bar{a})} = \frac{\Pr\left(c_w^b \middle| a\right)}{\Pr\left(\bar{t}_w\right)} \quad \text{by (25), (26), (27), (28), (29), transitivity.} \tag{30}$$

$$\frac{\Pr\left(\bar{t}_w\right)\Pr(b)}{\Pr(\bar{a})} = \Pr\left(c_w^b \middle| a\right) \quad \text{by (30), algebra.}$$

Therefore

$$\Pr\left(c_w^b \middle| a\right) = \frac{\Pr\left(\bar{t}_w\right)\Pr(b)}{\Pr(\bar{a})} \quad \text{if} \quad a \perp\!\!\!\perp \bar{t}_w \text{ and } \Pr\left(c_w^b \middle| a\bar{t}_w\right) = \frac{\Pr(b)}{\Pr(\bar{a})} \text{ and } b \to \bar{a}\,.$$

□

Proof of Theorem 7. Suppose $|A| = 2$ and $a \perp\!\!\!\perp t_w$. Then

$$\bar{a} \perp\!\!\!\perp \bar{t}_w \quad \text{by hypothesis.} \tag{31}$$

$$\Pr\left(\bar{a}c_w^a\right) = \Pr\left(\bar{a}\bar{t}_w\right) \quad \text{by (E4), hypothesis, (E13), (K4).} \tag{32}$$

$$\Pr\left(\bar{a}c_w^a\right) = \Pr(\bar{a})\Pr\left(\bar{t}_w\right) \quad \text{by (31), (32).} \tag{33}$$

$$\frac{\Pr\left(\bar{a}c_w^a\right)}{\Pr(\bar{a})} = \Pr\left(\bar{t}_w\right) \quad \text{by (33), algebra.} \tag{34}$$

$$\Pr\left(c_w^a | \bar{a}\right) = \Pr\left(\bar{t}_w\right) \quad \text{by (34), definition of conditional probability.}$$

Therefore

$$\Pr\left(c_w^a | \bar{a}\right) = \Pr\left(\bar{t}_w\right) \quad \text{if} \quad |A| = 2 \text{ and } a \perp\!\!\!\perp t_w\,.$$

□

Proof that (7) follows from (6) in Example 2. Consider this subexpression of (6):

$$\sum_{a \in A \setminus \{q\}} \prod_{w \in W} \frac{\Pr\left(\bar{t}_w\right)\Pr(a)}{\Pr(\bar{q})}\,.$$

Since a and q in this subexpression are rolls of a die and the prior probabilities of all rolls are equal, this is true:

$$\sum_{a \in A \setminus \{q\}} \prod_{w \in W} \frac{\Pr\left(\bar{t}_w\right)\Pr(a)}{\Pr(\bar{q})} = \sum_{a \in A \setminus \{q\}} \frac{\Pr(q)^{|W|}}{\Pr(\bar{q})^{|W|}} \prod_{w \in W} \Pr\left(\bar{t}_w\right)\,. \tag{35}$$

Since the content of the summation on the right-hand side of (35) is the same for any value of a,

$$\sum_{a\in A\setminus\{q\}} \frac{\Pr(q)^{|W|}}{\Pr(\bar{q})^{|W|}} \prod_{w\in W} \Pr(\bar{t}_w) = (|A|-1)\frac{\Pr(q)^{|W|}}{\Pr(\bar{q})^{|W|}} \prod_{w\in W} \Pr(\bar{t}_w)\ . \tag{36}$$

Since, in this scenario, prior probability is equally divided amongst alternatives in A,

$$\frac{\Pr(q)}{\Pr(\bar{q})} = \frac{\Pr(q)}{1-\Pr(q)} = \frac{\frac{1}{|A|}}{1-\frac{1}{|A|}} = \frac{\frac{1}{|A|}}{\frac{|A|-1}{|A|}} = \frac{1}{|A|-1}\,,$$

which, in combination with the laws of exponents, entails

$$\frac{\Pr(q)^{|W|}}{\Pr(\bar{q})^{|W|}} = \left(\frac{\Pr(q)}{\Pr(\bar{q})}\right)^{|W|} = \left(\frac{1}{|A|-1}\right)^{|W|} = (|A|-1)^{-|W|}\,. \tag{37}$$

Multiplying the leftmost and rightmost parts of (37) by $(|A|-1)\prod_{w\in W}\Pr(\bar{t}_w)$ yields

$$(|A|-1)\frac{\Pr(q)^{|W|}}{\Pr(\bar{q})^{|W|}} \prod_{w\in W} \Pr(\bar{t}_w) = (|A|-1)^{1-|W|} \prod_{w\in W} \Pr(\bar{t}_w) \tag{38}$$

and so by (35), (36), (38) and the transitive property of equality,

$$\sum_{a\in A\setminus\{q\}} \prod_{w\in W} \frac{\Pr(\bar{t}_w)\Pr(a)}{\Pr(\bar{q})} = (|A|-1)^{1-|W|} \prod_{w\in W} \Pr(\bar{t}_w)\ . \tag{39}$$

In (6), substituting the right-hand side of (39) for the left-hand side of (39) yields

$$\Pr\left(q \middle| \bigwedge_{w\in W} c_w^q\right) = \frac{\prod_{w\in W} \Pr(t_w)}{\prod_{w\in W} \Pr(t_w) + (|A|-1)^{1-|W|} \prod_{w\in W} \Pr(\bar{t}_w)}\,.$$

□

Proof that (8) follows from (10) in Example 3. The example supposes that each of W attest to q and $\bigwedge_{w\in W} q \perp\!\!\!\perp t_w$. With assumption (10), the scenario meets the antecedent conditions of Theorem 1. Since all witnesses attest to q, c_w^q can be substituted for $c_w^{T(w)}$ in Theorem 1, yielding

$$\Pr\left(q \middle| \bigwedge_{w\in W} c_w^q\right) = \frac{\Pr(q) \prod_{w\in W} \Pr(c_w^q|q)}{\sum_{a\in A} \Pr(a) \prod_{w\in W} \Pr(c_w^q|a)}\,. \tag{40}$$

Since $A = \{q, \bar{q}\}$, the sum in (40) can be expanded and therefore

$$\Pr\left(q \middle| \bigwedge_{w\in W} c_w^q\right) = \frac{\Pr(q) \prod_{w\in W} \Pr(c_w^q|q)}{\Pr(q) \prod_{w\in W} \Pr(c_w^q|q) + \Pr(\bar{q}) \prod_{w\in W} \Pr(c_w^q|\bar{q})}\,. \tag{41}$$

TABLE 1. Scenarios where one independence condition is met and another is not.

	#1	#2
$\Pr\left(qc_w^q c_x^q t_w t_x\right)$	0.49	0.54
$\Pr\left(qc_w^q \bar{c}_x^q t_w \bar{t}_x\right)$	0.15	0.18
$\Pr\left(q\bar{c}_w^q c_x^q \bar{t}_w t_x\right)$	0.15	0.06
$\Pr\left(q\bar{c}_w^q \bar{c}_x^q \bar{t}_w \bar{t}_x\right)$	0.01	0.02
$\Pr\left(\bar{q}c_w^q c_x^q \bar{t}_w \bar{t}_x\right)$	0.03	0.01
$\Pr\left(\bar{q}c_w^q \bar{c}_x^q \bar{t}_w t_x\right)$	0.01	0.01
$\Pr\left(\bar{q}\bar{c}_w^q c_x^q t_w \bar{t}_x\right)$	0.01	0.04
$\Pr\left(\bar{q}\bar{c}_w^q \bar{c}_x^q t_w t_x\right)$	0.15	0.14
$\Pr\left(q\right)$	0.80	0.80
$\Pr\left(qc_w^q\right)$	0.64	0.72
$\Pr\left(qc_x^q\right)$	0.64	0.60
$\Pr\left(qc_w^q c_x^q\right)$	0.49	0.54
$\Pr\left(t_w\right)$	0.80	0.90
$\Pr\left(t_x\right)$	0.80	0.75
$\Pr\left(t_w t_x\right)$	0.64	0.68
$\Pr\left(t_w\right)\Pr\left(t_x\right)$	$0.64 \cdot 0.64$	$0.90 \cdot 0.75$
$\Pr\left(q\right)\Pr\left(t_w\right)$	$0.80 \cdot 0.80$	$0.80 \cdot 0.90$
$\Pr\left(q\right)\Pr\left(t_x\right)$	$0.80 \cdot 0.80$	$0.80 \cdot 0.75$
$\Pr\left(c_w^q \mid q\right)\Pr\left(c_x^q \mid q\right)$	$\frac{0.64}{0.80} \cdot \frac{0.64}{0.80}$	$\frac{0.72}{0.80} \cdot \frac{0.60}{0.80}$
$\Pr\left(c_w^q c_x^q \mid q\right)$	$\frac{0.49}{0.80}$	$\frac{0.54}{0.80}$

Since $\bigwedge_{w \in W} q \perp\!\!\!\perp t_w$, the substitutions described in Theorem 5 and Theorem 7 are allowable for (41) and therefore

$$\Pr\left(q \middle| \bigwedge_{w \in W} c_w^q\right) = \frac{\Pr\left(q\right) \prod_{w \in W} \Pr\left(t_w\right)}{\Pr\left(q\right) \prod_{w \in W} \Pr\left(t_w\right) + \Pr\left(\bar{q}\right) \prod_{w \in W} \Pr\left(\bar{t}_w\right)}.$$

□

Proof that (9) *does not entail* (10) *and* (10) *does not entail* (9)*.* Consider scenarios where there is a condition q and two witnesses w and x who must testify for or against q. In these scenarios, $A = \{q, \bar{q}\}$ and $W = \{w, x\}$. If so, then there are 32 ways to combine the propositions q, c_w^q, c_x^q, t_w, t_x and their negations. Of these combinations, 24 describe impossible scenarios such as $qc_w^q c_x^q \bar{t}_w t_x$, where q is true and w claims q but w does not tell the truth. Since they are impossible, their probabilities are always 0. For the remaining 8 combinations, there exist nonzero probabilities that add up to 1 while making condition (9) true and condition (10) false. One such set of probabilities is shown in column #1 of Table 1. There also exist nonzero probabilities that add up to 1 while making condition (10) true and condition (9) false. One such set of probabilities is shown in column #2 of Table 1.

The first eight rows of Table 1 are probabilities of combinations of conditions. The next seven rows of Table 1 are probabilities that can be calculated by summing quantities in the first eight rows. The last five rows are the probabilities involved in (9) and (10), which can be derived from the middle rows using (K4) and the definitions of conditional probability and independence, noting that, per (E1) and (E2), qt_w is equivalent to qc_w^q and qt_x is equivalent to qc_x^q.

Since (9) can be true while (10) is false, (9) does not entail (10). Since (10) can be true while (9) is false, (10) does not entail (9). □

Proof that (11) *follows from* (10) *in Example 3 if* $\Pr(q) = \Pr(\bar{q})$. With assumption (10), the scenario meets the antecedent conditions for Corollary 1. Applying Corollary 1 to alternatives q and $\bar{q}$ yields

$$\frac{\Pr\left(q \middle| \bigwedge_{w\in W} c_w^{T(w)}\right)}{\Pr\left(\bar{q} \middle| \bigwedge_{w\in W} c_w^{T(w)}\right)} = \frac{\Pr(q)\prod_{w\in W}\Pr\left(c_w^{T(w)}|q\right)}{\Pr(\bar{q})\prod_{w\in W}\Pr\left(c_w^{T(w)}|\bar{q}\right)}. \tag{42}$$

Since by hypothesis $\Pr(q) = \Pr(\bar{q})$, these two terms cancel each other out, and

$$\frac{\Pr(q)\prod_{w\in W}\Pr\left(c_w^{T(w)}|q\right)}{\Pr(\bar{q})\prod_{w\in W}\Pr\left(c_w^{T(w)}|\bar{q}\right)} = \frac{\prod_{w\in W}\Pr\left(c_w^{T(w)}|q\right)}{\prod_{w\in W}\Pr\left(c_w^{T(w)}|\bar{q}\right)}. \tag{43}$$

Let function T, in addition to mapping each witness in W to the alternative in A to which the witness attested, also map each alternative in A to the set of witnesses in W which attested to it. With T, the products on the right-hand side of (43) can be split into two parts each, one for witnesses who claim q and another for witnesses who claim $\bar{q}$:

$$\frac{\prod_{w\in W}\Pr\left(c_w^{T(w)}|q\right)}{\prod_{w\in W}\Pr\left(c_w^{T(w)}|\bar{q}\right)} = \frac{\prod_{w\in T(q)}\Pr\left(c_w^{q}|q\right)\prod_{w\in T(\bar{q})}\Pr\left(c_w^{\bar{q}}|q\right)}{\prod_{w\in T(\bar{q})}\Pr\left(c_w^{\bar{q}}|\bar{q}\right)\prod_{w\in T(q)}\Pr\left(c_w^{q}|\bar{q}\right)}. \tag{44}$$

Since $\bigwedge_{w\in W} q \perp\!\!\!\perp t_w$, the substitutions described in Theorem 5 and Theorem 7 are allowable:

$$\frac{\prod_{w\in T(q)}\Pr\left(c_w^{q}|q\right)\prod_{w\in T(\bar{q})}\Pr\left(c_w^{\bar{q}}|q\right)}{\prod_{w\in T(\bar{q})}\Pr\left(c_w^{\bar{q}}|\bar{q}\right)\prod_{w\in T(q)}\Pr\left(c_w^{q}|\bar{q}\right)} = \frac{\prod_{w\in T(q)}\Pr(t_w)\prod_{w\in T(\bar{q})}\Pr(\bar{t}_w)}{\prod_{w\in T(\bar{q})}\Pr(t_w)\prod_{w\in T(q)}\Pr(\bar{t}_w)}. \tag{45}$$

Since all witnesses are equally credible,

$$\frac{\prod_{w\in T(q)}\Pr(t_w)\prod_{w\in T(\bar{q})}\Pr(\bar{t}_w)}{\prod_{w\in T(\bar{q})}\Pr(t_w)\prod_{w\in T(q)}\Pr(\bar{t}_w)} = \frac{\Pr(t_w)^{|T(q)|}\Pr(\bar{t}_w)^{|T(\bar{q})|}}{\Pr(t_w)^{|T(\bar{q})|}\Pr(\bar{t}_w)^{|T(q)|}}. \tag{46}$$

By the definitions of m and n given in the example,

$$\frac{\Pr(t_w)^{|T(q)|}\Pr(\bar{t}_w)^{|T(\bar{q})|}}{\Pr(t_w)^{|T(\bar{q})|}\Pr(\bar{t}_w)^{|T(q)|}} = \frac{\Pr(t_w)^{m}\Pr(\bar{t}_w)^{n}}{\Pr(t_w)^{n}\Pr(\bar{t}_w)^{m}}. \tag{47}$$

By the definition of complement,

$$\frac{\Pr(t_w)^m \Pr(\bar{t}_w)^n}{\Pr(t_w)^n \Pr(\bar{t}_w)^m} = \frac{\Pr(t_w)^m (1 - \Pr(\bar{t}_w))^n}{\Pr(t_w)^n (1 - \Pr(\bar{t}_w))^m}. \tag{48}$$

Therefore, by (42), (43), (44), (45), (46), (47), (48), the transitive property of equality and the laws of exponents,

$$\frac{\Pr\left(q \middle| \bigwedge_{w \in W} c_w^{T(w)}\right)}{\Pr\left(\bar{q} \middle| \bigwedge_{w \in W} c_w^{T(w)}\right)} = \frac{\Pr(t_w)^m (1 - \Pr(t_w))^n}{\Pr(t_w)^n (1 - \Pr(t_w))^m} = \frac{\Pr(t_w)^{m-n}}{(1 - \Pr(t_w))^{m-n}}.$$

□

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